NEW SOLUTIONS OF EINSTEIN EQUATIONS IN SPHERICAL SYMMETRY: THE COSMIC CENSOR TO THE COURT

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ABSTRACT. A new class of solutions of the Einstein field equations in spherical symmetry is found. The new solutions are mathematically described as the metrics admitting separation of variables in area-radius coordinates. Physically, they describe the gravitational collapse of a class of anisotropic elastic materials. Standard requirements of physical acceptability are satisfied, in particular, existence of an equation of state in closed form, weak energy condition, and existence of a regular Cauchy surface at which the collapse begins. The matter properties are generic in the sense that both the radial and the tangential stresses are non vanishing, and the kinematical properties are generic as well, since shear, expansion, and acceleration are also non-vanishing. As a test-bed for cosmic censorship, the nature of the future singularity forming at the center is analyzed as an existence problem for o.d.e. at a singular point using techniques based on comparison theorems, and the spectrum of endstates - blackholes or naked singularities - is found in full generality. Consequences of these results on the Cosmic Censorship conjecture are discussed.

1. INTRODUCTION

The final state of gravitational collapse is an important open issue of classical gravity. It is, in fact, commonly believed that a collapsing star that it is unable to radiate away - via e.g. supernova explosion - a sufficient amount of mass to fall below the neutron star limit, will certainly and inevitably form a black hole, so that the singularity corresponding to diverging values of energy and stresses will be safely hidden - at least to faraway observers - by an event horizon. However, this is nothing more than a conjecture - what Roger Penrose first called a "Cosmic Censorship" conjecture [24] - and has never been proved. On the contrary, in the last twenty years of research, many analytic examples of (spherically symmetric) naked singularities satisfying the principles of physical reasonableness have been discovered.

Spherically symmetric naked singularities can be divided into two groups: those occurring in scalar fields models [3, 4] and those occurring in astrophysical sources modeled with continuous media, which are of exclusive interest here (see [11] for a recent review). The first (shell focusing) examples of naked singularities where discovered numerically by Eardley and Smarr [9] and, analytically, by Christodoulou [2], in the gravitational collapse of dust clouds. Since then, the dust models have been developed in full details: today we know the complete spectrum of endstates of the gravitational collapse of spherical dust with arbitrary initial data [16]. This spectrum can be described as follows: given the initial density and velocity of the dust, a integer n can be introduced, in such a way that if n equals one or two, the singularity is naked, if $n \geq 4$ the singularity is covered, while if $n = 3$ the system

undergoes a transition form naked singularities to blackholes in dependence of the value of a certain parameter. If the collapse is marginally bound, the integer n is the order of the first non vanishing derivative of the energy density at the center.

The dust models can, of course, be strongly criticized from the physical point of view. Since from one side very few results are known on perfect (i.e. isotropic) fluids and, on the other side, stresses have to be expected to be anisotropic in strongly collapsed objects, one of us initiated some years ago a program whose objective was to understand what happens if the gravitational collapse occurs in presence of only one non-vanishing stress, the tangential one [20, 21]. The program can be said to have been completed: we know the complete spectrum of the gravitational collapse with tangential stresses in dependence of the data both for the clusters [10, 12] and for the continuous media models [8].

The results of this program are somewhat puzzling. In fact, what one should expect on physical grounds is that the equation of state plays a relevant role in deciding the final state of the collapse. Well, it is only *apparently* so. In fact, also in the tangential stress case a parameter n can be defined, in such a way that the endstates of collapse depend on n exactly in the same way as in dust.

In the present paper we present a complete, new model of gravitational collapse which includes both radial and tangential stresses. This is done deriving a class of anisotropic solutions which is in itself new, and contains the dust and the tangential stress metrics as special cases. As far as we are aware, this is the first time that the spectrum of the endstates of a solution satisfying all the requirements of physical reasonability and exhibiting both radial and tangential stresses is found.

As will be discussed in the concluding section, in view of the results of the present paper, the case for a cosmic censor - at least in spherical symmetry - becomes very weak.

2. Einstein Field Equations in area–radius coordinates

Consider a spherically symmetric collapsing object. The general line element in comoving coordinates can be written as

$$\mathrm{d}s^2 = -e^{2\nu}\mathrm{d}t^2 + (1/\eta)\,\mathrm{d}r^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{2.1}$$

(where ν, η and R are functions of r and t). We shall use a dot and a prime to denote derivatives with respect to t and r respectively.

In the present paper we consider as admissible sources of the gravitational field only those matter models which admit a well-defined thermodynamical description in terms of the standard relativistic mechanics of continua. We recall that the physical properties of an elastic material in isothermal conditions can be described in terms of one state function w, which depends on three parameters (the so called strain parameters) and on the coordinates. In the comoving description, the strain tensor of the material can be expressed in a "purely gravitational" way and, as a consequence, the state function depends only on the space-space part of the metric(see e.g. [19] and references therein). In addition, in spherical symmetry, the state function cannot depend on angles, and therefore the equation of state of a general spherically symmetric material can be given as a function of r and of two

strain parameters [20]:

$$w = w(r, R, \eta) \ . \tag{2.2}$$

It is useful also to introduce the matter density

$$\rho = \frac{\sqrt{\eta}}{4\pi E R^2} \tag{2.3}$$

(where $E = E(r)$ is an arbitrary positive function) so that the internal energy density is given by $\epsilon = \rho w$. If w depends only on ρ, one recovers the case of the barotropic perfect fluid; in general, however, the stresses are anisotropic and are given by the following stress-strain relations:

$$p_r = 2\rho\eta\frac{\partial w}{\partial \eta}, \qquad p_t = -\frac{1}{2}\rho R\frac{\partial w}{\partial R}. \tag{2.4}$$

Comoving coordinates r, t are extremely useful in dealing with gravitational collapse because of the transparent physical meaning of the comoving time. We shall, however, make systematic use here also of another system of coordinates, the so-called area-radius coordinates, which were first introduced by Ori [23] to study charged dust, and then successfully applied to other models of gravitational collapse (see e.g. [10, 21]). These coordinates prove extremely useful for technical purposes, as will be clear below. In these coordinates the comoving time is replaced by R. The velocity field of the material $v^\mu = e^{-\nu}\delta^\mu_t$ transforms to $v^\lambda = e^{-\nu}\dot{R}\delta^\lambda_R$ and therefore the transformed metric, although non diagonal, is still comoving. It can be written as

$$\mathrm{d}s^2 = -A\mathrm{d}r^2 - 2B\mathrm{d}R\mathrm{d}r - C\mathrm{d}R^2 + R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \ , \tag{2.5}$$

where A, B and C are functions of r and R. Obviously

$$C = \frac{1}{v_\lambda v^\lambda} := \frac{1}{u^2} \tag{2.6}$$

where we have denoted $u = |\dot{R}e^{-\nu}|$. Formula (2.2) implies that the internal energy w now depends on two coordinates (r and R) and on only one field variable η. It is convenient to introduce the quantity

$$\Delta := B^2 - AC = \frac{1}{\eta u^2} > 0 \ , \tag{2.7}$$

(so that $\sqrt{\eta} = 1/(u\sqrt{\Delta})$ and $B = -\sqrt{\Delta + A/u^2}$), and to use A, u and $\sqrt{\Delta}$ as the fundamental field variables. A convenient set of Einstein equations for these is $G^r_r = 8\pi T^r_r$, $G^R_r = 8\pi T^R_r$ and $G^r_R = 8\pi T^r_R$ [21]. Denoting partial derivatives with a comma, the first of these equations reads

$$\frac{1}{R^2}\left[1 - \frac{A}{\Delta} - R\left(\frac{A}{\Delta}\right)_{,R}\right] = -8\pi\, p_r \ , \tag{2.8}$$

so that A decouples from u and Δ:

$$A = \Delta\left[1 - \frac{2F(r)}{R} - \frac{2\Lambda}{R}\right], \tag{2.9}$$

where

$$\Lambda := -\int_{R_0(r)}^{R} 4\pi\sigma^2 p_r(r,\sigma,\eta(r,\sigma))\,\mathrm{d}\sigma. \tag{2.10}$$

In the above formulae, the arbitrary positive function $R_0(r)$ describes the values of R on the initial data and will be conveniently taken to be

$$R_0(r) = r. \tag{2.11}$$

To find the physical meaning of the other arbitrary function F, we introduce the so called *Misner-Sharp mass* $\Psi(r,R)$. It is defined in such a way that the equation $R = 2\Psi$ spans the boundary of the *trapped region*, i.e. the region in which outgoing null rays reconverge. This boundary is the *apparent horizon* of the spacetime, and it can be shown that (see e.g. [14]):

$$\Psi(r,R) = (R/2)\left(1 - g^{\mu\nu}\partial_\mu R\,\partial_\nu R\right). \tag{2.12}$$

Transforming to radius-area coordinates and using (2.9), one easily gets

$$\Psi = (R/2)\left(1 - A/\Delta\right) = F(r) + \Lambda \tag{2.13}$$

and therefore the function F is the value of the Misner-Sharp mass on the data. Since Λ vanishes if the radial stress is zero, when $p_r = 0$ the Misner-Sharp mass is conserved during the evolution (i.e. independent on R). This is the main technical reason which makes the spacetimes with vanishing radial stresses simpler with respect to general collapse models.

Using (2.9) it can be shown that the two remaining field equations can be written as follows:

$$E\,\Psi_{,r} = \left(w + 2\eta\frac{\partial w}{\partial\eta}\right)\sqrt{u^2 + 1 - \frac{2\Psi}{R}} \tag{2.14}$$

and

$$(\sqrt{\Delta})_{,R} = -\frac{1}{\sqrt{u^2 + 1 - \frac{2\Psi}{R}}}\left(\frac{1}{u}\right)_{,r}. \tag{2.15}$$

3. The general solution admitting separation of variables

3.1. **The solution.** Let us consider the system of two coupled PDE's for Δ and u (2.14)–(2.15). If Λ does not depend on the field variables, equation (2.14) becomes algebraic and the system decouples. Functional properties of Λ are related to the choice of w and in fact, Λ will be independent of η if p_r does. This obviously happens if p_r is zero (leading to the already well known cases of dust and of vanishing radial stresses, for which w does not depend on η) but also if p_r is some function of r and R only. This in turn occurs if w is of the form

$$w(r,R,\eta) = h(r,R) + \frac{1}{\sqrt{\eta}}\,l(r,R), \tag{3.1}$$

the case $p_r = 0$ corresponding to $l = 0$. If l is non zero the radial pressure is non-vanishing

$$p_r = -\frac{1}{4\pi E R^2}\,l(r,R) \tag{3.2}$$

so that

$$\Lambda = \frac{1}{E}\int_r^R l(r,\sigma)\,\mathrm{d}\sigma. \tag{3.3}$$

Using (3.1) and (2.13), equation (2.14) gives

$$E\,\Psi_{,r} = h\sqrt{u^2 + 1 - \frac{2\Psi}{R}},$$

which allows us to compute the quantity u^2:

$$u^2(r,R) = -1 + \frac{2\Psi}{R} + \left(\frac{E\Psi_{,r}}{h}\right)^2. \tag{3.4}$$

As a consequence, (2.15) becomes a quadrature which allows the calculation of $\sqrt{\Delta}$:

$$\sqrt{\Delta} = -\int \frac{1}{Y(r,\sigma)}\frac{\partial\mathcal{H}}{\partial r}(r,\sigma)\,\mathrm{d}\sigma, \tag{3.5}$$

where we have defined

$$\mathcal{H}(r,R) = \sqrt{\frac{R}{2\Psi(r,R) + R(Y^2(r,R) - 1)}}, \tag{3.6}$$

$$Y(r,R) = \frac{E(r)\Psi_{,r}(r,R)}{h(r,R)}. \tag{3.7}$$

It is convenient to eliminate the indefinite integral in (3.5). For this aim, it is useful the following equation

$$Y(r,R) = R'(r,t(r,R))\sqrt{\eta}(r,t(r,R)), \tag{3.8}$$

that can be easily found from the relations between the metrics coefficients in the comoving and the area–radius coordinates. Using (2.7), (3.8) and the fact that $R' = 1$ for $R = r$, we get $\sqrt{\Delta}(r,r) = \frac{\mathcal{H}(r,r)}{Y(r,r)}$, which fixes the integration constant, to obtain

$$\sqrt{\Delta}(r,R) = \int_R^r \frac{1}{Y(r,\sigma)}\frac{\partial\mathcal{H}}{\partial r}(r,\sigma)\,\mathrm{d}\sigma + \frac{\mathcal{H}(r,r)}{Y(r,r)}. \tag{3.9}$$

3.2. **Physical requirements.** In this section we will discuss the conditions that have to be imposed on the equation of state and on the data, in order for the solutions to be physically meaningful.

From now on, we consider only spacetimes whose matter source satisfies the equation of state (3.1). For such spacetimes, using (2.13) and (3.3), the constitutive function w may be written as

$$w(r,R,\eta) = E(r)\left(\frac{\Psi_{,r}(r,R)}{Y(r,R)} + \frac{\Psi_{,R}(r,R)}{\sqrt{\eta}}\right), \tag{3.10}$$

therefore the energy density $\epsilon = \rho w$ and the stresses (2.4) are given by:

$$\epsilon = \frac{1}{4\pi R^2}\left(\frac{\sqrt{\eta}}{Y}\Psi_{,r} + \Psi_{,R}\right), \tag{3.11}$$

$$p_r = -\frac{\Psi_{,R}}{4\pi R^2}, \tag{3.12}$$

$$p_t = -\frac{\sqrt{\eta}}{8\pi R}\left(\frac{\Psi_{,rR}}{Y} - \frac{\Psi_{,r}}{Y^2}\frac{\partial Y}{\partial R} + \frac{\Psi_{,RR}}{\sqrt{\eta}}\right), \tag{3.13}$$

From now on we choose as independent arbitrary functions Ψ and Y. In the present paper, we assume C^k regularity of the data: the equation of state and the arbitrary functions are assumed to be Taylor-expandable up to the required order. Due to eq. (3.11), (3.12) and (3.13), it will then be possible to translate all the conditions to which the metrics have to satisfy in order to be physically meaningful (like e.g. weak energy condition or regularity of the center up to singularity formation, see below), in terms of conditions on the functions Ψ and Y or on their derivatives.

We first impose the weak energy condition (w.e.c.). In spherical symmetry, this condition is equivalent to $\epsilon \geq 0$, $\epsilon + p_r \geq 0$, $\epsilon + p_t \geq 0$. These inequalities lead to

$$\Psi_{,r} \geq 0, \quad \Psi_{,R} \geq 0, \quad \Psi_{,r} \geq \frac{R}{2}Y\left(\frac{\Psi_{,r}}{Y}\right)_{,R}, \quad \Psi_{,R} \geq \frac{R}{2}\Psi_{,RR}. \tag{3.14}$$

For an explicit solution satisfying weak energy condition we refer to the example in 3.5 below.

We also impose regularity of the metric at the center ('local flatness'). In comoving coordinates $r,\, t$ this amounts to require

$$R(0,t) = 0, \qquad e^{\lambda(0,t)} = R'(0,t). \tag{3.15}$$

In addition, the stress tensor must be isotropic at $r = 0$, that is

$$p_r(0,t) = p_t(0,t), \tag{3.16}$$

for any regular $t = \mathrm{const}$ hypersurface. Finally, we require the existence of a regular Cauchy surface ($t = 0$, say) carrying the initial data for the fields. These requirements are fundamental, since they assure that the singularities eventually forming will be a genuine outcome of the dynamics.

We have already chosen $R = r$ at $t = 0$; using (3.11), (3.8) and the fact that $R'(r,0) = 1$ we find the expression for the initial energy density $\epsilon_0(r)$:

$$\epsilon_0(r) = \frac{\Psi_{,r}(r,r) + \Psi_{,R}(r,r)}{4\pi r^2}. \tag{3.17}$$

Once the regularity conditions are satisfied, the data will be regular if this function is regular. For physical reasonability we also require the initial density to be decreasing outwards, that is $\epsilon_0'(r) \leq 0$ for $r > 0$. It is easy to check, that the above stated conditions are equivalent, in terms of the functions Ψ and Y, to the

following:

(3.18)

$$Y(0,0) = 1,$$

(3.19)

$$\Psi(0,0) = \Psi_{,r}(0,0) = \Psi_{,R}(0,0) = \Psi_{,rr}(0,0) = \Psi_{,rR}(0,0) = \Psi_{,RR}(0,0) = 0,$$

(3.20)

$$\Psi_{,rr}(r,r) + 2\Psi_{,rR}(r,r) + \Psi_{,RR}(r,r) - \frac{2}{r}\left(\Psi_{,r}(r,r) + \Psi_{,R}(r,r)\right) < 0.$$

Definition 3.1. We say that a spacetime is a physically valid area-radius separable spacetime (ARS) if

(1) the equation of state of the matter is (3.10), where $\Psi(r,R)$ and $Y(r,R)$ are arbitrary positive functions;
(2) the weak energy condition, the regularity condition, and the condition of decreasing initial density hold, i.e. $\Psi(r,R)$ and $Y(r,R)$ satisfy (3.14) and (3.18)–(3.20).

3.3. **Kinematics.** Spherically symmetric non-static solutions can be invariantly classified in terms of their kinematical properties (see e.g. [17, 18]). Most of the known solutions have vanishing shear, or expansion, or acceleration. For the solutions studied here all such parameters are generally non-vanishing.

The acceleration in comoving coordinates r, t is given by $a_\mu = \nu' \delta^r_\mu$, therefore, it can be uniquely characterized by the scalar $A := \sqrt{a_\mu a^\mu}$. Using (3.8) together with the relation

$$Y\nu' = R' Y_{,R} \tag{3.21}$$

(which easily follows from the the field equation in comoving coordinates $\dot{R}' = \dot{R}\nu' + R'\dot{\lambda}$) we get

$$A = Y_{,R}. \tag{3.22}$$

The expansion Θ is given by

$$\Theta = \pm u \left[\frac{(u\sqrt{\Delta})_{,R}}{u\sqrt{\Delta}} + \frac{2}{R}\right]. \tag{3.23}$$

Finally, the shear tensor σ^μ_ν can be uniquely characterized by the scalar $\sigma := (2/3)\sqrt{\sigma^\mu_\nu \sigma^\nu_\mu}$ given by

$$\sigma = \pm \frac{u}{3}\left[\frac{1}{R} - \frac{(u\sqrt{\Delta})_{,R}}{u\sqrt{\Delta}}\right]. \tag{3.24}$$

3.4. **Special classes.** Special cases of the solutions discussed above are:

(1) The dust (Tolman-Bondi) spacetimes. The energy density equals the matter density, and this implies $\Psi_{,R} = 0$ and $Y = E\Psi_{,r}$.
(2) The general solution with vanishing radial stresses. Vanishing of p_r implies $\Psi_{,R} = 0$ (see (3.12)), while Y depends also on R (if $Y_{,R} = 0$ one recovers dust). The properties of these solutions have been widely discussed in [20].

(3) An interesting new subclass is obtained by imposing the vanishing of the acceleration. This subclass corresponds to a choice of the function Y depending on r only. It is well known that acceleration-free perfect fluid models can describe only very special collapsing objects (the pressure must be a function of the comoving time only) while these anisotropic solutions exhibit several interesting features, like e.g. a complete spectrum of endstates [7].

3.5. **An explicit example: "anisotropysations" of Tolman–Bondi–de Sitter spacetime.** The investigation of various explicit examples of new solutions within the class presented here, as well as their physical applications, will be presented in a forthcoming paper [7]. We restrict ourself here only to sketch a simple acceleration free (i.e. $Y = Y(r)$) case, essentially with the aim of showing explicitly that the new sector of the solutions (w.r. to dust and tangential stress case) is non empty.

A simple way to produce explicit examples of new collapsing solutions within the equation of state considered here is to choose the mass function as the sum of a function of r only (governing the dust limit) and a function of R only (governing the anisotropic stresses). In order to satisfy the requirements of Definition 3.1, easy calculations show that the function Ψ must have the form

$$\Psi(r,R) = \int_0^r \phi(s)s^2\,\mathrm{d}s + \int_0^R \chi(\sigma)\sigma^2\,\mathrm{d}\sigma, \tag{3.25}$$

where χ and ϕ are positive and non increasing functions. In the particular case in which χ is a constant, the contribution of χ to the energy-momentum tensor is formally the same as that of a cosmological constant, and therefore the spacetime coincides with the so called Tolman–Bondi–de Sitter (TBdS), describing the collapse of spherical dust. This is actually the unique model of gravitational collapse in presence of a lambda term which is known so far. In recent years, the description of gravitational collapse in such spacetimes attracted a renewed interest both from the astrophysical point of view, since recent observations of high-redshift type Ia supernovae suggest a non-vanishing value of lambda, and from the theoretical point of view, after the proposal of the so-called Ads-Cft correspondence in string theory. In such a context, the solutions (3.25) can be used to investigate the effects of stresses by adding higher order terms to the function χ. For instance, choosing

$$\Psi(r,R) = \alpha r^3 + \frac{\Lambda R^3}{1 + R/R_*}$$

where R_* is a positive constant, one obtains a model which is TBdS homogeneous and isotropic near the center but becomes anisotropic and inhomogeneous when R is of the order of R_* (we stress, however, that this is only one possible application of this special sub-class of solutions).

4. Conditions for singularity formation

4.1. **Shell crossing and shell-focusing singularities.** Due to eqs.(2.7) and (3.11), the energy density becomes singular if, during the evolution, R or $u\sqrt{\Delta}$ vanish. The latter case corresponds to $R'(r,t) = 0$ in comoving coordinates and is called a *shell–crossing* singularity, since it is generated by shells of matter intersecting each other. Shell-crossing singularities correspond to weak (in Tipler sense) divergences

of the invariants of the Riemann tensor. As a consequence, these singularities are usually considered as ”not interesting”, although no proof of extensibility is as yet present in the literature. In any case, we shall concentrate here on the the shell-focusing $R = 0$ singularity, for which no kind of extension is possible. Therefore, we work out the conditions ensuring that crossing singularities do not occur.

Since we are considering a collapsing scenario, $R < r$ during evolution. Therefore, due to eq. (3.9), it is sufficient to require strict positivity of the function $\sqrt{\Delta}(r, 0)$ together with non-increasing behavior of $\sqrt{\Delta}(r, R)$ w.r. to R. We thus have the following:

Proposition 4.1. *In a ARS spacetime the formation of shell–crossing singularities does not occur if*

$$\Psi_{,r}(r, R) + Y(r, R)\, Y_{,r}(r, R)\, R \geq 0 \quad \textit{for} \;\; r \geq 0,\; R \in [0, r], \tag{4.1}$$

$$\int_0^r \frac{1}{Y(r,\sigma)} \frac{\partial \mathcal{H}}{\partial r}(r, \sigma)\, d\sigma + \frac{\mathcal{H}(r, r)}{Y(r, r)} > 0 \qquad \textit{for} \;\; r > 0. \tag{4.2}$$

The above condition, although being only sufficient, allows for a wide class of new metrics. It becomes also necessary in the particular case of dust [13].

Spherically symmetric matter models do not, of course, unavoidably form singularities if stresses are present. One can, in fact, construct models of oscillating or bouncing spheres. In comoving coordinates (r, t), the locus of the zeroes of $R(r, t)$ - if any - defines implicitly a singularity curve $t_s(r)$ via $R(r, t_s(r)) = 0$. The quantity $t_s(r)$ represents the comoving time at which the shell labelled r becomes singular, and therefore the central singularity forms if

$$\lim_{r \to 0^+} t_s(r) = t_*, \tag{4.3}$$

is finite (t_* is positive due to the regularity of the data at $t = 0$). To study the behavior of this limit in dependence of the choice of the data, we use $\dot{R} = -e^{\nu} u$. Integrating along a flow line we get

$$t = \int_R^r e^{-\nu(r,\sigma)} \mathcal{H}(r, \sigma) \mathrm{d}\sigma, \tag{4.4}$$

where the initial condition (2.11) has been used. Up to time reparameterizations we can assume $\nu(0, t) = 0$. Using this fact, together with (3.21), it can be seen that ν is uniformly bounded and $\lim_{r \to 0^+} \nu(r, \sigma) = 0$ uniformly for $\sigma \in [0, r]$. Therefore it is necessary and sufficient to require that, for r near 0, $\mathcal{H}(r, \sigma)$ is integrable with respect to σ in $[0, r]$ and

$$0 < \lim_{r \to 0^+} \int_0^r \mathcal{H}(r, \sigma) d\sigma < \infty. \tag{4.5}$$

For this aim, let us consider the following Taylor expansion centered at the point $(0, 0)$, where we take Definition 3.1 into account:

(4.6)

$$2\Psi(r, R) + R(Y^2(r, R) - 1) = 2Y_{,r}(0, 0) r\, R + 2Y_{,R}(0, 0) R^2 + \sum_{k \geq 3} \sum_{i+j=k} h_{ij} r^i R^j.$$

Changing variable $\sigma = r\tau$ and recalling the definition of $\mathcal{H}$ (eq. (3.6)) the integral in (4.5) becomes

$$\int_0^1 \frac{\sqrt{\tau}\,\sqrt{r}}{\sqrt{2Y_{,r}(0,0)\tau + 2Y_{,R}(0,0)\tau^2 + r\,\sum_{i+j=3} h_{ij}\tau^j + r\,\varphi(r)}}\,\mathrm{d}\tau,$$

with $\varphi(0) = 0$. Convergence of this integral to a finite non-zero value requires vanishing of the zero order terms and at least one non vanishing first order term in the denominator (i.e. $(h_{30}, h_{21}, h_{12}) \neq (0,0,0)$). We shall, however, consider only the sub-case in which h_{30} (which we will call α hereafter) is non-vanishing, since a vanishing α would correspond to a bad-behaved "dust limit" of the solutions, i.e. when the source of the anisotropic stress becomes very weak. For the same reason, α cannot be negative otherwise the weak energy condition would be violated in the same limit. Thus:

Proposition 4.2. *In an ARS spacetime shell focusing singularities form if*

$$Y_{,r}(0,0) = Y_{,R}(0,0) = 0,\ \alpha > 0. \tag{4.7}$$

From the above discussion, we now define the subclass of ARS spacetimes that we are dealing with hereafter.

Definition 4.3. An ARS spacetime is called collapsing if:

(1) Shell-crossing singularities do not form (i.e. Ψ and Y satisfy to (4.1), (4.2));
(2) Shell-focusing singularities form in a finite amount of comoving time (i.e. Ψ and Y satisfy to (4.7)).

4.2. **The apparent horizon and the nature of the possible singularities.** A key role in the study of the nature of a singularity is played by the apparent horizon (see for instance [14]). The apparent horizon in comoving coordinates is the curve $t_h(r)$ defined by $R(r, t_h(r)) = 2\Psi(r, R(r, t_h(r)))$. In area-radius coordinates, one has correspondingly a curve $R_h(r)$ and, since $\Psi_{,R}(0,0) = 0$, implicit function theorem ensures that $R_h(r)$ is well defined in a right neighborhood of $r = 0$ and such that $R_h(r) < r$. This fact, along with the requirement $\Psi_{,R} \geq 0$ coming from the w.e.c. (3.14) ensures $\Psi(r, R_h(r)) < \Psi(r, r)$. Now using (4.4) it is

$$0 \leq t_s(r) - t_h(r) = \int_0^{2\Psi(r,R_h)} e^{-\nu}\mathcal{H}(r,\sigma)d\sigma \leq \int_0^{\Psi(r,r)} e^{-\nu}\mathcal{H}(r,\sigma)d\sigma \tag{4.8}$$

Changing variable $\sigma = r\tau$ as before and using the fact that ν is bounded and $\Psi(0,0) = 0$, we find that this integral converges to 0 as r tends to 0, ensuring the following

Proposition 4.4. *In a collapsing ARS spacetime the center becomes trapped at the same comoving time at which it becomes singular, that is*

$$\lim_{r\to 0^+} t_h(r) = t_* = \lim_{r\to 0^+} t_s(r). \tag{4.9}$$

In the models studied in this paper, the only singularity that can be naked is the central ($r = 0$) one. Indeed, a singularity cannot be naked if it occurs after the formation of the apparent horizon $t_h(r)$. But using (4.8), we get $t_s(r) = t_h(r)$ only if $\Psi(r, R_h(r)) = 0$, which happens if and only if $r = 0$. Thus, the shell

labeled $r > 0$ becomes trapped before becoming singular, and hence all non–central singularities are censored. An important consequence of this fact is that, although the area-radius coordinates map the singularity curve to the axis $R = 0$, it is actually only the 'point' $R = r = 0$ that has to be analyzed in order to establish the causal structure of the spacetime.

Next section will be devoted to the study of the nature of the central singularity. A singularity is either *locally naked*, if it is visible to nearby observers, *globally naked*, if it is visible also to far-away observers, or *censored*, if it is invisible to any observer. We shall not be concerned here with the issue of global nakedness, so that our test of cosmic censorship will be performed on the *strong* cosmic censorship hypothesis: each singularity is invisible to any observer. We call a blackhole a singularity of this kind, although one could conceive situations in which the singularities are locally naked but hidden by a global event horizon, an obvious example of this being of course the Kerr spacetime with mass to angular momentum per unit mass ratio greater than one. However, in all the examples so far discovered of *dynamical* formation of naked singularities, one can easily attach smoothly to the region of spacetime which contains the naked singularity a regular asymptotic region containing no event horizon. It is, therefore, likely that the unique version of the cosmic censorship conjecture that can be the object of a mathematical proof is the strong one.

5. THE SPECTRUM OF ENDSTATES

5.1. **The radial null-geodesic equation.** The equation of radial null geodesics in the coordinate system (r, R) is given by

$$\frac{\mathrm{d}R}{\mathrm{d}r} = u\sqrt{\Delta}[Y - u]. \tag{5.1}$$

Indeed, all null curves in two dimensions can be reparameterized to become geodesics. Therefore, (5.1) comes from (2.5) setting $\mathrm{d}s^2 = 0$, $(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) = 0$, and requiring the future–pointing character of the curve.

The center $R = r = 0$ is (locally) naked if there exists a future pointing local solution $R_g(r)$ of the geodesic equation which extends back to the singularity (i.e. $R(0) = 0$) and "escapes from the apparent horizon", that is $R_g(r) > R_h(r)$ for $r > 0$. We will study in full details only the existence of *radial* null geodesics emanating from the singularity. We are, however, going to prove that if a singularity is radially censored (that is, no radial null geodesics escape), then it is censored (see subsection 5.5 below).

In what follows we shall need to consider *sub* and *super* solutions of the equation (5.1). We recall that a function $y_0(r)$ is said to be a subsolution (respectively supersolution) of an ordinary differential equation of the kind $y' = f(r, y)$ if it satisfies $y_0' \leq f(r, y_0)$ (respectively $\geq$).

5.2. **The main theorem.** It is easy to check that in a collapsing ARS spacetime (see Definition 4.3) the Taylor expansion of $\sqrt{\Delta}(r, 0)$ at the center is given by

$$\sqrt{\Delta}(r, 0) = \xi\, r^{n-1} + \dots, \tag{5.2}$$

where ξ is a positive number and n is a positive integer. The following theorem shows that the causal nature of such spacetimes is fully governed by these two quantities.

Theorem 5.1. *In a collapsing ARS spacetime, the singularity forming at the center is locally naked if $n = 1$, if $n = 2$, or if $n = 3$ and $\frac{\xi}{\alpha} > \xi_c$ where*

$$\xi_c = \frac{26 + 15\sqrt{3}}{2}. \tag{5.3}$$

Otherwise the singularity is covered.

For sake of clarity, we divide the proof of the main theorem into two parts.

5.3. Sufficient conditions for existence.

Theorem 5.2. *In a collapsing ARS spacetime, the singularity forming at the center is locally naked if $n < 3$ or if $n = 3$ and $\frac{\xi}{\alpha} > \xi_c$.*

To prove the theorem we first need the following crucial result:

Lemma 5.3. *In a collapsing ARS spacetime the apparent horizon is a supersolution of equation* (5.1).

Proof. This result can actually be proved for much more general spacetimes (essentially, only the weak energy condition is needed) [6]. However, we give here a very simple proof for the model at hand. As we have seen, we always suppose that α is positive (see Definition 4.3). This implies $2\Psi(r, 0) \cong \alpha r^3$, and from the definition of the apparent horizon $R_h(r)$, it is

$$R_h(r) = 2\Psi(r, R_h(r)) = 2\Psi(r, 0) + R_h(r)\Psi_{,R}(r, s_r) \cong \alpha r^3, \tag{5.4}$$

$(s_r \in (0, R_h(r)))$ since $\Psi_{,R}(r, \xi_r)$ is infinitesimal. This implies $R_h'(r) \cong 3\alpha r^2 > 0$ in an open right neighborhood of 0, whereas it is easily seen that $u(r, R_h(r)) = Y(r, R_h(r))$, implying that the right hand side of (5.1) gives zero when evaluated at (r, R_h). Hence $R_h(r)$ is a supersolution of (5.1). □

Lemma 5.4. *The singularity forming at the center is naked if there exists a subsolution of equation* (5.1) *of the form $\tilde{R}(r) = xr^3$, where $x > \alpha$.*

Proof. The singularity is naked if there exists a geodesic $R_g(r)$ such that $R_g(r) > R_h(r)$ in an open right neighborhood of 0. If $\tilde{R}(r) = xr^3$ is a subsolution of (5.1) and $x > \alpha$, then $\tilde{R}(r) > R_h(r)$. Let $r_0 > 0$ and $R_g(r)$ a geodesic through $R_g(r_0) \in]R_h(r_0), \tilde{R}(r_0)[$. This curve cannot cross the subsolution from below, neither it can cross the supersolution from above. Thus $R_g(r)$ is defined in $]0, r_0]$ and $\lim_{r\to 0^+} R_g(r) = 0$. Therefore R_g is the sought geodesic. □

Theorem 5.2. We will derive sufficient conditions for $\tilde{R}(r) = xr^3$ to be a subsolution of (5.1) with $x > \alpha$; the result will then follow from Lemma 5.4. We have

$$u(r, \tilde{R}(r)) \cong \sqrt{\frac{\alpha}{x}} \tag{5.5}$$

(see (3.6)), and

$$Y(r,\tilde{R}(r)) - u(r,\tilde{R}(r)) \cong 1 - \sqrt{\frac{\alpha}{x}}, \tag{5.6}$$

since $Y(0,0)=1$.

Now, from (3.9),

$$\sqrt{\Delta}(r,\tilde{R}(r)) = \sqrt{\Delta}(r,0) - \int_0^{xr^3} \frac{1}{Y(r,\sigma)}\frac{\partial \mathcal{H}}{\partial r}(r,\sigma)\,\mathrm{d}\sigma. \tag{5.7}$$

But

$$\int_0^{xr^3} \frac{1}{Y(r,\sigma)}\frac{\partial \mathcal{H}}{\partial r}(r,\sigma)\,\mathrm{d}\sigma = -\int_0^{xr^3} \frac{\sqrt{\sigma}\left[\sum_{i+j=3} i\, h_{ij} r^{i-1}\sigma^j + \dots\right]}{2Y(r,\sigma)\left[\sum_{i+j=3} h_{ij} r^i \sigma^j + \dots\right]^{3/2}}\,\mathrm{d}\sigma \cong$$

$$\cong -\int_0^{x} \frac{r^{3/2}\sqrt{\tau}\left[\sum_{i+j=3} i\, h_{ij} r^{i-1+3j}\tau^j\right]}{2\left[\sum_{i+j=3} h_{ij} r^{i+3j}\tau^j\right]^{3/2}}\, r^3\mathrm{d}\tau \cong -\frac{r^2}{\sqrt{\alpha}}\tau^{3/2}|_{\tau=0}^{\tau=x} = -\sqrt{\frac{x}{\alpha}}xr^2$$

where the variable change $\sigma = r^3\tau$ has been performed, and we recall that $\alpha = h_{30}$. Then

$$\sqrt{\Delta}(r,\tilde{R}(r)) \cong \xi\, r^{n-1} + \sqrt{\frac{x}{\alpha}}xr^2. \tag{5.8}$$

The curve $\tilde{R}$ is certainly a subsolution of (5.1) if the following inequality holds:

$$3\,x\,r^2 < \left(1-\sqrt{\frac{\alpha}{x}}\right)\left(\sqrt{\frac{\alpha}{x}}\xi\, r^{n-1} + xr^2\right). \tag{5.9}$$

This inequality holds always if $n=1$ or if $n=2$; namely, the term on the right hand side is a positive function that behaves like r^m whenever $x>\alpha$. If $n=3$ condition (5.9) is equivalent to

$$S(x,\frac{\xi}{3}) < 0, \tag{5.10}$$

where

$$S(x,p) \equiv 2x^2 + \sqrt{\alpha}x^{3/2} - 3p\,\sqrt{\alpha}\,x^{1/2} + 3p\,\alpha, \tag{5.11}$$

and using standard techniques it can be seen that (5.10) holds for some $x > \alpha$ if and only if

$$\frac{\xi}{\alpha} > \frac{26+15\sqrt{3}}{2} = \xi_c\,. \tag{5.12}$$

□

5.4. **Necessary conditions for existence.**

Theorem 5.5. *In an collapsing ARS spacetime, if the singularity forming at the center is locally naked then* $n < 3$ *or,* $n = 3$ *and* $\frac{\xi}{\alpha} \geq \xi_c$*.*

To show the theorem, essentially adapting an argument exploited in [5] for dust solutions, we need the following Lemma.

Lemma 5.6. *In a collapsing ARS spacetime, if a curve* $\tilde{R}(r)$ *is a geodesic emanating from the central singularity such that*

$$t_{\tilde{R}}(r) := \int_{\tilde{R}(r)}^{r} e^{-\nu}\mathcal{H}(r,\sigma)\,d\sigma$$

verifies $\lim_{r\to 0^+} t_{\tilde{R}}(r) = t_0 = \lim_{r\to 0^+} t_s(r)$*, then*

$$\lim_{r\to 0^+} \frac{\tilde{R}(r)}{r} = 0. \tag{5.13}$$

Proof. We have $\lim_{r\to 0^+}(t_s(r) - t_{\tilde{R}}(r)) = 0$. Recalling that

$$t_s(r) = \int_0^r e^{-\nu}\mathcal{H}(r,\sigma)\,\mathrm{d}\sigma,$$

we also have

$$t_s(r) \;-\; t_{\tilde{R}}(r) \;=\; \int_0^{\tilde{R}(r)} e^{-\nu}\mathcal{H}(r,\sigma)\,\mathrm{d}\sigma \;\cong\; \int_0^{\tilde{R}(r)/r} \frac{e^{-\nu}\,\sqrt{\tau}}{\left(\sum_{i+j=3} h_{ij}\tau^j\right)^{1/2}}\,\mathrm{d}\tau,$$

and since ν is bounded this quantity must be infinitesimal, so that (5.13) must hold. □

Theorem 5.5. Let $\tilde{R}(r) = x(r)r^3$ be a geodesic such that $x(r) > \alpha$, and $\tilde{R}(0) = 0$. Using Lemma 5.6, $x(r)r^2$ must be infinitesimal, so it is a straightforward calculation to verify that

$$u(r,\tilde{R}(r)) \cong \sqrt{\frac{\alpha}{x(r)}}, \tag{5.14}$$

$$Y(r,\tilde{R}(r)) - u(r,\tilde{R}(r)) \cong 1 - \sqrt{\frac{\alpha}{x(r)}}, \tag{5.15}$$

whereas, using the same arguments as in (5.8), it is

$$\begin{aligned}\sqrt{\Delta}(r,\tilde{R}(r)) = \sqrt{\Delta}(r,0) - \int_0^{x(r)r^3} \frac{1}{Y(r,\sigma)}\frac{\partial\mathcal{H}}{\partial r}(r,\sigma)\,\mathrm{d}\sigma \cong \\ \cong \xi\, r^{n-1} + \sqrt{\frac{x(r)}{\alpha}}x(r)r^2.\end{aligned} \tag{5.16}$$

Since $\tilde{R}(r)$ is a geodesic, (5.1) yields

$$x'(r)r \cong \left(1 - \sqrt{\frac{\alpha}{x(r)}}\right)\left[\sqrt{\frac{\alpha}{x(r)}}\xi r^{n-3} + x(r)\right] - 3\,x(r). \tag{5.17}$$

By contradiction, let us first assume $n > 3$. Therefore, if $x(r)$ went to $+\infty$ for $r \to 0^+$, using (5.17) it would be

$$x'(r) = -\frac{2}{r} x(r)\psi(r) + \varphi(r),$$

where $\varphi(r)$ is a bounded function and $\psi(0) = 1$. Using comparison theorems for ODE's there would exist $\lambda > 0$ such that $x(r) \geq \frac{\lambda}{r^2}$ and so $\tilde{R}(r) \geq \lambda r$, which is in contradiction with Lemma 5.6. If $\lim_{r\to 0^+} x(r) = l$, with $l \in (0, +\infty)$, then a straightforward calculation of the limit of both sides in (5.17) yields a contradiction as well, because $x'(r) \cong -\frac{k\,x(r)}{r}$ for some positive constant k. If $x(r)$ does not have limit, there exists a sequence r_m of local minima for $x(r)$ such that $r_m \to 0$ for $m \to \infty$, $x'(r_m) = 0$ and $x(r_m)$ is bounded. Evaluating (5.17) for $r = r_m$, and taking the limit of both sides for $m \to \infty$ we get a contradiction once again. Then $n \leq 3$. In the case $n = 3$, equation (5.17) can be written as

$$x'(r)r = -\frac{S(x(r), \frac{\xi}{3})}{x(r)} + \ldots, \tag{5.18}$$

where $S(x, p)$ was defined in (5.11). Arguing as before, $x(r)$ cannot go to $+\infty$ as r goes to 0 since $\tilde{R}(r)$ would be bounded from below by λr for some $\lambda > 0$. If $\lim_{r\to 0^+} x(r) = l < +\infty$ then $S(x(r), \frac{\xi}{3}) \to S(l, \frac{\xi}{3})$ and, from (5.18), $S(l, \frac{\xi}{3}) = 0$. Since $l \geq \alpha$, equation (5.12) implies $\frac{\xi}{\alpha} \geq \xi_c$. If $x(r)$ does not have limit, as before there exists a sequence $r_m \to 0$ of local minima for $x(r)$ such that $x'(r_m) = 0$ and $x(r_m)$ is bounded, and this yields, using (5.18), that $S(x(r_m), \frac{\xi}{3}) \to 0$, that is $x(r_m)$ converges to a root of $S(x, \frac{\xi}{3}) = 0$ greater that α, which is possible again only if $\frac{\xi}{\alpha} \geq \xi_c$. □

5.5. **Non-radial geodesics.** We have limited our analysis to radial null geodesics. However, we will now show that, if no radial null geodesic escapes from the singularity, then no null geodesic escapes at all. In other words, we show that a radially censored singularity is censored (in the case of dust spacetimes, this result was first given by Nolan and Mena [22]).

Let by contradiction $\tilde{R}(r)$ be a non radial null geodesics escaping from the center, that we will suppose radially censored. Arguing in a similar way as for recovering (5.1), we have that $\tilde{R}(r)$ solves equation

$$\frac{\mathrm{d}\tilde{R}}{\mathrm{d}r} = -u^2 \left[B + \sqrt{\Delta + CL^2/R^2}\right], \tag{5.19}$$

where L^2 is the conserved angular momentum. Since C is positive we have

$$\frac{\mathrm{d}\tilde{R}}{\mathrm{d}r} \leq -u^2 \left[B + \sqrt{\Delta}\right],$$

that is $\tilde{R}(r)$ is a subsolution of the null radial geodesic equation (5.1). By hypothesis $\tilde{R}(r) > R_h(r)$ for $r > 0$, and $\tilde{R}(0) = R_h(0)$. Then a comparison argument in ODE similar to the one exploited in Lemma 5.4 ensures the existence of a radial null geodesic, which is a contradiction. Thus, the following holds true:

Proposition 5.7. *Any singularity radially censored is censored.*

We emphasize that this is a general result, not depending on the class of solutions we are dealing with, but only on the spherical symmetry of the model.

5.6. **Physical interpretation of the results.** As we have seen, although the mathematical structure of the solutions of the Einstein field equations and the way in which this structure governs the properties of the differential equation of radial geodesics is extremely intricate, our final results are nevertheless extremely simple: what governs the whole machinery is just the first term of the Taylor expansion of $\sqrt{\Delta}(r,0)$ near the center (Theorem 5.1). To understand what this results says in physical term, it is convenient to write $\sqrt{\Delta}(r,0)$ as

$$\sqrt{\Delta}(r,0) = I_1(r) + I_2(r) \tag{5.20}$$

where (see also (3.9))

$$I_1(r) := \frac{1}{Y(r,r)}\left[\frac{\partial}{\partial r}\int_0^r \mathcal{H}(r,\sigma)\,\mathrm{d}\sigma\right], \quad I_2(r) := \int_0^r \gamma(r,\sigma)\frac{\partial \mathcal{H}}{\partial r}(r,\sigma)\,\mathrm{d}\sigma, \tag{5.21}$$

and we have defined

$$\gamma(r,\sigma) := \frac{1}{Y(r,\sigma)} - \frac{1}{Y(r,r)}.$$

The function $I_1(r)$ has the following behavior:

$$\begin{aligned} I_1(r) = \frac{1}{Y(r,r)}\frac{\partial}{\partial r}\left(\int_0^1 \mathcal{H}(r,r\tau)r\,\mathrm{d}\tau\right) &\cong \\ \cong \int_0^1 \frac{\sqrt{\tau}(p\,r^{p-1}Q(\tau)+\dots}{2\left[P(\tau)-r^pQ(\tau)+\dots\right]^{3/2}} &\cong p\,a\,r^{p-1}+\dots, \end{aligned} \tag{5.22}$$

where the following Taylor expansion through $(0,0)$ has been introduced:

$$2\Psi(r,R) + R(Y^2(r,R)-1) = \sum_{i+j=3} h_{ij}r^jR^j + \sum_{i+j=3+p} h_{ij}r^jR^j + \dots, \tag{5.23}$$

and

$$P(\tau) = \sum_{i+j=3} h_{ij}\tau^j, \quad Q(\tau) = -\sum_{i+j=3+p} h_{ij}\tau^j, \quad a = \int_0^1 \frac{Q(\tau)\sqrt{\tau}}{2P(\tau)^{3/2}}\,\mathrm{d}\tau. \tag{5.24}$$

The remaining summand $I_2(r)$ is zero if Y depends only on r, i.e. if the acceleration $A(r,R) = Y_{,R}(r,R)$ (formula (3.22)) vanishes. Its behavior is as follows:

$$\begin{aligned} I_2(r) = -\int_0^r \gamma(r,\sigma)\frac{\sqrt{\sigma}\left[\sum_{i+j=3} i\,h_{ij}r^{i-1}\sigma^j+\dots\right]}{2\left[\sum_{i+j=3} h_{ij}r^i\sigma^j+\dots\right]^{3/2}}\,\mathrm{d}\sigma &= \\ = -\int_0^1 \gamma(r,r\tau)\frac{\sqrt{\tau}\left[\sum_{i+j=3} i\,h_{ij}\tau^j+\dots\right]}{2r\left[P(\tau)+\dots\right]^{3/2}}\,\mathrm{d}\tau&. \end{aligned}$$

Now, using Taylor expansion of Y through the center

$$Y(r,R) = \varphi(r) + \sum_{\substack{i+j=q+1\\ j>0}} k_{ij}r^iR^j + \dots,$$

($\varphi(r)$ contains the terms where R does not appear, and $\varphi(0) = 1$), it is

$$\gamma(r, r\tau) \cong Y(r,r) - Y(r, r\tau) = r^{1+q} \sum_{\substack{i+j=q+1 \\ j>0}} k_{ij}(1-\tau^j) + \ldots = S(\tau)\, r^{1+q} + \ldots,$$

where q is easily seen to be the order of the first non vanishing term of the expansion of the acceleration near the center. Thus, we obtain:

$$I_2(r) = -b\, r^q + \ldots, \quad \text{where } b := \int_0^1 S(\tau) \frac{\sqrt{\tau}\left[\sum_{i+j=3} i\, h_{ij} \tau^j\right]}{2P(\tau)^{3/2}}\, \mathrm{d}\tau.$$

It follows that

$$\sqrt{\Delta}(r, 0) = p\, a\, r^{p-1} - b\, r^q + \ldots \tag{5.25}$$

The index n defined in formula (5.2) is thus the smaller[1] between p and $q+1$. The value of p is clearly related to the degree of inhomogeneity of the system, since one can generate a low value of it taking a low order of non-vanishing derivatives of the initial density profile at the center (formula (3.17)). This effect can be related to the shear as well (see e.g. [1, 15]). The value of q, and thus the second term, is related strictly to the acceleration: it vanishes if the acceleration vanishes, and in any case it does not contributes to the nature of the final state if the system "does not accelerate enough" near the center. The effect of this term on naked singularities formation can be considered as a new feature of our models. It can, in fact, be shown that (virtually) *all* the particular cases already known in literature of formation of naked singularities in the gravitational collapse of continua (for instance the dust and the tangential stress model) can be retrieved as particular cases of our main theorem, and that in all such cases the acceleration term is negligible and the effect does not occur (of course, these cases to not exhaust the content of Theorem 5.1).

6. Discussion and conclusions

From the very beginning, the problem of Cosmic Censorship was shown to be linked to specific mathematical (as opposed to physically transparent) properties of the available arbitrary functions, like e.g. 'first derivative of initial density equal to zero, second non-zero at the center'. In all the cases which have been discovered so far the situation became more and more intricate. In the present paper, we have constructed a new class of solutions which contains as subcases all the solutions for which censorship has already been investigated in full details. The new solutions add to the (rather scarce) set of spherically symmetric spacetimes whose kinematical properties are generic. The analysis of the structure of their singularities allowed us to show that a general and simple pattern actually exists. This pattern follows from Theorem 5.1: given any set of regular data, and any equation of state within the considered class, the formation of naked singularities or blackholes depends on a sort of selection mechanism. The mechanism works as follows: it extrapolates the value of an integer n and selects the final state according to it. The extrapolation essentially depends on the kinematical invariants of the motion. If the resulting n equals one or two or it is greater than three, the final state is decided and has no

[1]One can conceive very special cases in which the terms exactly balance each other and the index n has to be defined at the next order.

other dependence on the data or on the matter properties. Dimensional quantities such as e.g. value of the derivative of the density at the center, cosmological constant, value of the derivative of the velocity profile at the center, profile of the state equation for tangential stresses, and so on, play a further role only at the transition between the two endstates, occurring at $n = 3$. This role is to combine themselves to produce a non-dimensional quantity which acts as a critical parameter.

The (spherically symmetric) cosmic censor seems to answer to the court, that the formation of naked singularities or blackholes is essentially a *local and kinematical* phenomenon: it neither depends (or weakly depends) on *what is collapsing*, nor it depends on the details of the data characterizing *how it starts collapsing*. The formation or whatsoever of blackholes or visible singularities depends only on the kinematical properties of the motion near the center of symmetry of the system.

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